

MEASURING CLUB-SEQUENCES WITH LARGE CONTINUUM

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ABSTRACT. *Measuring*, as defined by J. Moore, says that for every sequence $(C_\delta)_{\delta < \omega_1}$ with each C_δ being a closed subset of δ there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$, a tail of $C \cap \delta$ is either contained in or disjoint from C_δ . We answer a question of Moore by building a forcing extension satisfying measuring together with $2^{\aleph_0} > \aleph_2$. The construction works over any model of ZFC and can be described as a finite support forcing iteration with systems of countable structures as side conditions and with symmetry constraints.

1. INTRODUCTION

One of the most frustrating problems faced by set theorists working with iterated proper forcing is the lack of techniques for producing models in which the continuum has size greater than the second uncountable cardinal. In this paper we solve this problem in the specific case of measuring, a very strong negation of Club Guessing at ω_1 introduced by Justin Moore (see [3]). This work is a natural continuation of our previous work in [1] (see also [2]), where we showed $2^{\aleph_0} > \aleph_2$ to be consistent together with a number of consequences of the Proper Forcing Axiom (PFA).

Our approach in [1] consisted in building, starting from CH, a certain type of finite support forcing iteration of length κ (in a general sense of ‘forcing iteration’) using what one may describe as finite ‘symmetric systems’ of countable elementary substructures of a fixed $H(\kappa)^1$ as side conditions. These systems of structures were added at the first stage of the iteration. Roughly speaking, the fact that the supports of the conditions in the iteration were finite ensured that the inductive proofs of the relevant facts – mainly that the iteration has the \aleph_2 -chain

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¹This κ is exactly the value that 2^{\aleph_0} attains at the end of the construction.

condition and that it is proper – went through. The use of the sets of structures as side conditions was crucial in the proof of properness.²

In the present paper we add a higher degree of ‘local’ symmetry in the ‘single step’ forcing notions involved and use it to build a model of measuring.

Definition 1.1. (Moore, [3]) *Measuring* is the following statement: Let $\mathcal{C} = (C_\delta)_{\delta < \omega_1}$ be such that each C_δ is a closed subset of δ (where δ is endowed with the order topology). Then there is a club $C \subseteq \omega_1$ which *measures* C_δ for every $\delta \in C$. Specifically, this means that for every $\delta \in C$ there is some $\alpha < \delta$ with either $(C \cap \delta) \setminus \alpha \subseteq C_\delta$ or $(C \setminus \alpha) \cap C_\delta = \emptyset$. We will also say that C *measures* \mathcal{C} .

Measuring is of course equivalent to its restriction to club-sequences, where $(C_\delta)_{\delta < \omega_1}$ is a club-sequence if every C_δ is a club of δ . Also, measuring clearly implies that for every ladder system $(C_\delta)_{\delta \in \text{Lim} \cap \omega_1}$ there is a club $C \subseteq \omega_1$ such that $C \cap C_\delta$ is finite for all $\delta \in C$,³ and that for every sequence $(f_\alpha)_{\alpha \in \omega_1}$, if each f_α is a continuous function from α into ω , then there is a club $C \subseteq \omega_1$ such that $f_\delta \restriction C \neq \omega$ for every $\delta \in C$.⁴ Finally, it is easily seen to follow from PFA, and even from its bounded form BPFA.⁵

Our main theorem is the following.

Theorem 1.2. (CH) *Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is a proper poset \mathcal{P} with the \aleph_2 -chain condition such that the following statements hold after forcing with \mathcal{P} .*

- (1) $2^{\aleph_0} = \kappa$
- (2) *Measuring*

The rest of the paper is devoted to proving Theorems 1.2, and is organized as follows: Section 2 starts with the central notion of symmetric system of structures. We then proceed, in Subsection 2.1, to the definition of a sequence $(\mathcal{P}_\alpha)_{\alpha \leq \kappa}$ of partial orders (\mathcal{P}_κ will be shown to witness Theorem 1.2). In Section 3 we give the basic analysis of our construction; in particular, we prove that it has the \aleph_2 -chain condition, its properness, that \mathcal{P}_α embeds completely in \mathcal{P}_β if $\alpha < \beta \leq \kappa$, and

²For more on the motivation for this type of construction see [1] and [2].

³ $(C_\delta)_{\delta \in \text{Lim} \cap \omega_1}$ is a ladder system if each C_δ is a cofinal subset of δ of order type ω . The above statement is the negation of what is usually called *Weak Club Guessing* (for ω_1).

⁴This is the negation of a statement Moore calls \bar{U} (mho) (see [6]).

⁵The proof involves a forcing for adding a suitable club with side conditions. The ‘single-step forcing’ in our construction is a variation of this type of forcing.

that \mathcal{P}_κ forces $2^{\aleph_0} = \kappa$ (Lemmas 3.3, 3.7, 3.2 and 3.10). These results, together with the final lemma (Lemma 3.11), establish Theorem 1.2.

For the most part our notation follows set-theoretic standards as set forth for example in [4] and in [5]. If N is a set whose intersection with ω_1 is an ordinal, then δ_N will denote this intersection. If N is a set, \mathbb{P} is a partial order and G is a (V -generic) filter of \mathbb{P} , $N[G]$ will denote $\{\tau_G : \tau \in N, \tau \text{ a } \mathbb{P}\text{-name}\}$, where τ_G denotes the interpretation of τ by G . Also, G is N -generic if $G \cap A \cap N \neq \emptyset$ for every maximal antichain A of \mathbb{P} belonging to N . A condition p in \mathbb{P} is (N, \mathbb{P}) generic if p forces that \dot{G} is N -generic in the above sense. Note that we are *not* assuming that \mathbb{P} is in N in any of these two sentences.

If T is a predicate (i.e., a subset) of some $H(\theta)$ and $\mathcal{N} = \langle N, T \cap N \rangle$ is a substructure of $\langle H(\theta), T \rangle$, we also denote \mathcal{N} by $\langle N, T \rangle$.⁶ Sets N will often be identified with the structure $\langle N, \in \rangle$. Recall that the elementary diagram of a structure $\langle N, T \rangle$ is the collection of sentences with parameters holding in $\langle N, T \rangle$.

Finally, we will consider the following natural notion of rank: Given two sets \mathcal{N}, N , we define the *Cantor–Bendixson rank of N with respect to \mathcal{N}* , $rank(\mathcal{N}, N)$, by specifying that $rank(\mathcal{N}, N) \geq 1$ if and only if for every $a \in N$ there is some $M \in \mathcal{N} \cap N$ such that $a \in M$ and, for each ordinal $\mu \geq 1$, that $rank(\mathcal{N}, N) > \mu$ if and only if for every $a \in N$ there is some $M \in \mathcal{N} \cap N$ such that $a \in M$ and $rank(\mathcal{N}, M) \geq \mu$.⁷

2. PROVING THEOREM 1.2: THE CONSTRUCTION

The proof of Theorem 1.2 will be given in a sequence of lemmas in this and the next section.

Let $\kappa \geq \omega_2$ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$, and let $\Phi : \kappa \rightarrow H(\kappa)$ be a surjection such that for every x in $H(\kappa)$, $\Phi^{-1}(\{x\})$ is unbounded in κ .

As anticipated in the introduction, \mathcal{P} will be the final member \mathcal{P}_κ of a certain sequence $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ of forcing notions. Together with $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ we will also define a sequence $\langle T^\alpha : \alpha < \kappa \rangle$ of subsets of $H(\kappa)$ ⁸ and a corresponding sequence $\langle \mathcal{M}^\alpha : \alpha < \kappa \rangle$ of clubs of $[H(\kappa)]^{\aleph_0}$.

⁶For example, $\langle N, \in \rangle$ will denote the structure $\langle N, \in \cap N \times N \rangle$.

⁷Note that if X is a set of ordinals and δ is an ordinal, then $rank(X, \delta) \geq 1$ if and only if δ is a limit point of ordinals in X and, for each ordinal $\mu \geq 1$, $rank(X, \delta) > \mu$ if and only if δ is a limit of ordinals ϵ with $rank(X, \epsilon) \geq \mu$.

⁸We will see the T^α 's as truth predicates.

2.0.1. *Symmetric systems of structures.* One central notion in our construction will be that of ‘symmetric system of structures’. We start by defining what we mean by this.

Definition 2.1. Let χ be an uncountable cardinal, \vec{P} a sequence $(P^\xi)_{\xi < \beta}$ of subsets of $H(\chi)$ (for some ordinal β), \mathcal{M} a club of $[H(\chi)]^{\aleph_0}$, and \mathcal{N} a finite set. We will say that \mathcal{N} is a \vec{P} -symmetric \mathcal{M} -system if the following conditions hold.

- (α) $\mathcal{N} \subseteq \mathcal{M}$.
- (β) For all $N, N' \in \mathcal{N}$ and all $\xi \in N \cap \beta$, if $\delta_N = \delta_{N'}$ and $\xi' := \Psi_{N,N'}(\xi) < \beta$, then there is a unique isomorphism $\Psi_{N,N'}$ between $\langle N, P^\xi \rangle$ and $\langle N', P^{\xi'} \rangle$.
Furthermore, we ask that $\Psi_{N,N'}$ be the identity on $\chi \cap N \cap N'$.
- (γ) For all N_0, N_1 in \mathcal{N} , if $\delta_{N_0} < \delta_{N_1}$, then there is some $N_2 \in \mathcal{N}$ such that $\delta_{N_2} = \delta_{N_1}$ and $N_0 \in N_2$.
- (δ) For all N_0, N_1, N_2 in \mathcal{N} , if $N_0 \in N_1$ and $\delta_{N_1} = \delta_{N_2}$, then $\Psi_{N_1,N_2}(N_0) \in \mathcal{N}$.

We may omit mentioning suitable parameters \vec{P} and \mathcal{M} when they are not relevant. If $H(\chi)$ is understood or irrelevant we call \mathcal{N} a *symmetric system (of structures)*.

Throughout this paper, if N and N' are such that there is a unique isomorphism from N into N' , then we denote this isomorphism by $\Psi_{N,N'}$. The following facts are easy consequences from the above definition.

Fact 2.2. Let $\chi, \vec{P} = (P^\xi)_{\xi < \beta}$ and \mathcal{M} be as in Definition 2.1. Let \mathcal{N} be a \vec{P} -symmetric \mathcal{M} -system, let $N, N' \in \mathcal{N}$ be such that $\delta_N = \delta_{N'}$, and let ξ, ξ' be ordinals in β such that $\xi \in N$ and $\Psi_{N,N'}(\xi) = \xi'$. Then $\Psi_{N,N'}$ is an isomorphism between the structures $\langle N, \in, P^\xi, \mathcal{N} \cap N \rangle$ and $\langle N', \in, P^{\xi'}, \mathcal{N} \cap N' \rangle$.

Fact 2.3. Let $\vec{P} = (P^\xi)_{\xi < \beta}$ and \mathcal{M} be as in Definition 2.1. Let $\mathcal{N}_0 = \{N_i^0 : i < m\}$ and $\mathcal{N}_1 = \{N_i^1 : i < m\}$ be \vec{P} -symmetric \mathcal{M} -systems of structures. Suppose that $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = X$, that there is an isomorphism

$$\Psi : \langle \bigcup_{i < m} N_i^0, \in, X \rangle \longrightarrow \langle \bigcup_{i < m} N_i^1, \in, X \rangle$$

fixing X , and that for all $\xi \in \beta \cap \bigcup_{i < m} N_i^0$, if $\xi' = \Psi(\xi) \in \beta$, then $\langle \bigcup_{i < m} N_i^0, \in, P^\xi, X, N_i^0 \rangle_{i < m}$ and $\langle \bigcup_{i < m} N_i^1, \in, P^{\xi'}, X, N_i^1 \rangle_{i < m}$ are isomorphic structures. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a \vec{P} -symmetric \mathcal{M} -system of structures.

Proof. The proof is a routine verification. Let us show for example that if $i_0, i_1 < m$ are such that $\delta_{N_{i_0}^0} = \delta_{N_{i_1}^1}$, then $\Psi_{N_{i_0}^0, N_{i_1}^1}$ fixes $Ord \cap N_{i_0}^0 \cap N_{i_1}^1$: Let Ψ be the isomorphism between $\langle \bigcup_{i < m} N_i^0, \in, X, N_i^0 \rangle_{i < m}$ and $\langle \bigcup_{i < m} N_i^1, \in, X, N_i^1 \rangle_{i < m}$. If $\gamma \in Ord \cap N_{i_0}^0 \cap N_{i_1}^1$, then $\gamma \in X \cap N_{i_0}^0$, which implies that $\Psi(\gamma) = \gamma \in N_{i_0}^1 \cap N_{i_1}^1$. But then $\gamma \in N_{i_0}^0 \cap N_{i_1}^1$ as Ψ is an isomorphism between the structures $\langle \bigcup_{i < m} N_i^0, \in, X, N_i^0 \rangle_{i < m}$ and $\langle \bigcup_{i < m} N_i^1, \in, X, N_i^1 \rangle_{i < m}$, which implies that $\Psi_{N_{i_0}^0, N_{i_1}^1}(\gamma) = \gamma$ and hence that $(\Psi \upharpoonright N_{i_1}^0 \circ \Psi_{N_{i_0}^0, N_{i_1}^1})(\gamma) = \Psi_{N_{i_0}^0, N_{i_1}^1}(\gamma) = \gamma$. \square

2.1. The definition of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$. Now we proceed to the definition of $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ and $\langle T^\alpha : \alpha < \kappa \rangle$.

For every $\alpha < \kappa$, \mathcal{M}^α will be the set of $N \in [H(\kappa)]^{\aleph_0}$ such that $\langle N, T^\alpha \rangle \preceq \langle H(\kappa), T^\alpha \rangle$. We let $T^0 \subseteq H(\kappa)$ code Φ together with the restriction of \in to $H(\kappa)$. For every nonzero $\beta < \kappa$, if T^α has been defined for all $\alpha < \beta$, we let $\vec{T}^\beta = (T^\alpha)_{\alpha < \beta}$ and let $T^\beta \subseteq H(\kappa)$ code the elementary diagram of $\langle H(\kappa), T^\alpha \rangle_{\alpha < \beta}$.⁹ As we will see, each \mathcal{P}_α will be lightface definable in the structure $\langle H(\kappa), T^{\alpha+1} \rangle$. We let also $\mathcal{M}_*^\beta = \{N \in [H(\kappa)]^{\aleph_0} : N \in \bigcap_{\xi \in N \cap \beta} \mathcal{M}^\xi\}$.

We start with the definition of \mathcal{P}_0 : A condition in \mathcal{P}_0 will be a pair (\emptyset, Δ) , where

- (A1) Δ is a countable set of pairs of the form $(N, 0)$.
- (A2) $dom(\Delta)$ is a T^0 -symmetric \mathcal{M}^0 -system of countable substructures of $H(\kappa)$.

Given \mathcal{P}_0 -conditions $q_\epsilon = (\emptyset, \Delta_\epsilon)$ for $\epsilon \in \{0, 1\}$, q_1 extends q_0 if and only if

- (B) $\Delta_0 \subseteq \Delta_1$

In the definition of \mathcal{P}_0 -condition we have used 0 twice in a completely vacuous way. These (vacuous) 0's are there to ensure that the (uniformly defined) operation of restricting a condition in a (further) \mathcal{P}_β to an ordinal $\alpha < \beta$ yields a condition in \mathcal{P}_0 when applied to any condition in any \mathcal{P}_β and to $\alpha = 0$.

Given $\alpha \leq \kappa$ for which \mathcal{P}_α has been defined, let \dot{G}_α be a canonical \mathcal{P}_α -name for the generic object and let $\mathcal{N}_{\dot{G}_\alpha}$ be a canonical \mathcal{P}_α -name for the set $\bigcup \{\Delta_r^{-1}(\alpha) : r \in \dot{G}_\alpha\}$.¹⁰

Let $\beta < \kappa$, $\beta \neq 0$, and suppose that for all $\alpha < \beta$

- (o) we have defined T^α and \mathcal{P}_α , and

⁹We will actually make use of T^β only for successor β .

¹⁰As we will see, conditions in \mathcal{P}_α will be pairs $r = (s, \Delta_r)$, with Δ_r a set of pairs (N, γ) , γ an ordinal.

- (o) $\mathcal{P}_\alpha \subseteq H(\kappa)$ is a partial order with the \aleph_2 -chain condition consisting of pairs $r = (s, \Delta_r)$, with Δ_r a set of pairs of the form (N, γ) , with γ an ordinal.

The \aleph_2 -c.c. of \mathcal{P}_α will follow from Lemma 3.3.

We will use a certain poset in $V^{\mathcal{P}_\alpha}$ for measuring a given club-sequence.

2.1.1. *A forcing notion for measuring a club-sequence.* Suppose \dot{C} is a \mathcal{P}_α -name for a club-sequence $(C_\delta)_{\delta < \omega_1}$. We define next a forcing $\Theta_{\dot{C}}$, in $V^{\mathcal{P}_\alpha}$, for adding a club measuring \dot{C} :

Conditions in $\Theta_{\dot{C}}$ are triples (f, b, \mathcal{O}) with the following properties.

- (1) $\mathcal{O} \subseteq \mathcal{N}_{\dot{G}_\alpha}$ is a $\vec{T}^{\alpha+2}$ -symmetric $\mathcal{M}_*^{\alpha+2}$ -system of structures.
- (2) f is a finite function that can be extended to a normal function $F : \omega_1 \rightarrow \omega_1$ such that $F(\omega) = \omega$. Moreover, for every $\nu \in \text{dom}(f) \setminus (\omega + 1)$,
 - (2.1) $f(\nu) \in \{\delta_N : N \in \mathcal{O}\}$, and
 - (2.2) $\text{rank}(\mathcal{N}_{\dot{G}_\alpha} \cap \mathcal{M}_*^{\alpha+2}, N) \geq \nu$ for every $N \in \mathcal{O}$ such that $f(\nu) = \delta_N$.
- (3) b is a function with domain included in $\text{dom}(f) \setminus (\omega + 1)$. Moreover, the following holds for all $\nu \in \text{dom}(b)$:
 - (3.1) $b(\nu) < \nu$ and $b(\nu) + 1 \in \text{dom}(f)$.
 - (3.2) If $\nu_0 \in \text{dom}(f)$ is such that $b(\nu) < \nu_0 < \nu$, then $f(\nu_0) \notin C_{f(\nu)}$. Furthermore, if $\nu_1 \in \text{dom}(f)$ is such that $\nu_0 + 1 < \nu_1 < \nu$, then $[f(\nu_0), f(\nu_1)] \cap C_{f(\nu)} = \emptyset$.
 - (3.3) $\text{rank}(\{M \in \mathcal{N}_{\dot{G}_\alpha} \cap \mathcal{M}_*^{\alpha+2} : \delta_M \notin C_{f(\nu)}\}, N) \geq \nu$ for every $N \in \mathcal{O}$ such that $f(\nu) = \delta_N$.

Given $\Theta_{\dot{C}}$ -conditions $(f_0, b_0, \mathcal{O}_0)$ and $(f_1, b_1, \mathcal{O}_1)$, $(f_1, b_1, \mathcal{O}_1)$ extends $(f_0, b_0, \mathcal{O}_0)$ in case $f_0 \subseteq f_1$, $b_0 \subseteq b_1$, and $\mathcal{O}_0 \subseteq \mathcal{O}_1$.

The forcing $\Theta_{\dot{C}}$ is meant to add a club C measuring \dot{C} . This club is the range of the union of all functions f coming from conditions in the generic filter. The fact that this function is continuous and has domain ω_1 is ensured essentially by condition (2.2) in our definition. The function b represents the commitment to avoid a certain member C_δ of \dot{C} on a tail of $C \cap \delta$ for every δ in its domain. This, together with the conditions that must hold in case we make this commitment, is expressed in condition (3). By density, every ν will eventually be in the domain of a relevant b – in other words, we will promise that C avoids a tail of $f(\nu)$ – unless we cannot keep that promise. If we cannot keep that promise at a given ν , then a density argument using essentially condition (2) and the symmetry of \mathcal{O} will show that a tail

of $C \cap f(\nu)$ will automatically go into $C_{f(\nu)}$ (see the proof of Lemma 3.11).

The following lemma is immediate.

Lemma 2.4. *Suppose \dot{C} is a \mathcal{P}_α -name for a club-sequence. If (f, b, \mathcal{O}_0) and (f, b, \mathcal{O}_1) are conditions in $\Theta_{\dot{C}}$ and $\mathcal{O}_0 \cup \mathcal{O}_1$ is a $\vec{T}^{\alpha+2}$ -symmetric $\mathcal{M}_*^{\alpha+2}$ -system, then $(f, b, \mathcal{O}_0 \cup \mathcal{O}_1)$ is a condition in $\Theta_{\dot{C}}$ stronger than both (f, b, \mathcal{O}_0) and (f, b, \mathcal{O}_1) .*

2.1.2. *Resuming the construction.* We are now in a position to define \mathcal{P}_β in general for any $\beta > 0$, $\beta \leq \kappa$ (assuming \mathcal{P}_α defined for all $\alpha < \beta$).

If $\alpha < \kappa$ and \mathcal{P}_α is defined, we let $\Phi^*(\alpha)$ be a \mathcal{P}_α -name for (say) the sequence $(\delta)_{\delta < \omega_1}$ if $\Phi(\alpha)$ is not a \mathcal{P}_α -name for a club-sequence, and let $\Phi^*(\alpha)$ be $\Phi(\alpha)$ if $\Phi(\alpha)$ is a \mathcal{P}_α -name for a club-sequence.¹¹

Assume first that $\beta < \kappa$. Conditions in \mathcal{P}_β are pairs of the form $q = (p, \Delta)$ with the following properties.

(C0) p is a finite function such that $\text{dom}(p) \subseteq \beta$ and Δ is a set of pairs (N, γ) with $\gamma \leq \beta$.

(C1) $\Delta^{-1}(\beta)$ is a $\vec{T}^{\beta+1}$ -symmetric $\mathcal{M}_*^{\beta+1}$ -system.

(C2) For every $\alpha < \beta$, the restriction of q to α is a condition in \mathcal{P}_α .

This restriction is defined as

$$q|_\alpha := (p \upharpoonright \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\})$$

(C3) If $\alpha \in \text{dom}(p)$, then $p(\alpha)$ is of the form $\langle f^{p,\alpha}, b^{p,\alpha}, \mathcal{O}^{p,\alpha} \rangle$ and

$$(C3.1) \quad \Delta_{q|_{\alpha+1}}^{-1}(\alpha+1) \cap \mathcal{M}_*^{\alpha+2} \subseteq \mathcal{O}^{p,\alpha} \subseteq \Delta_{q|_\alpha}^{-1}(\alpha),$$

$$(C3.2) \quad \mathcal{P}_\alpha \upharpoonright q|_\alpha \text{ forces that } \langle f^{p,\alpha}, b^{p,\alpha}, \mathcal{O}^{p,\alpha} \rangle \text{ is a } \Theta_{\Phi^*(\alpha)}\text{-condition, and}$$

$$(C3.3) \quad \text{for all } N \in \Delta_{q|_{\alpha+1}}^{-1}(\alpha+1), \text{ if } \alpha \in N, \text{ then } \delta_N \in \text{dom}(f^{p,\alpha}) \text{ and } f^{p,\alpha}(\delta_N) = \delta_N.$$

Given conditions

$$q_\epsilon = (p_\epsilon, \Delta_\epsilon)$$

(for $\epsilon \in \{0, 1\}$) in \mathcal{P}_β , we will say that $q_1 \leq_\beta q_0$ if and only if the following holds.

(D1) $q_1|_\alpha \leq_\alpha q_0|_\alpha$ for all $\alpha < \beta$,

(D2) $\text{dom}(p_0) \subseteq \text{dom}(p_1)$

(D3) $f^{p_0,\alpha} \subseteq f^{p_1,\alpha}$, $b^{p_0,\alpha} \subseteq b^{p_1,\alpha}$ and $\mathcal{O}^{p_0,\alpha} \subseteq \mathcal{O}^{p_1,\alpha}$ for all $\alpha \in \text{dom}(p_0)$.¹²

(D4) $\Delta_0^{-1}(\beta) \subseteq \Delta_1^{-1}(\beta)$.

¹¹It will follow from our definition that, for all $\alpha < \beta$, a \mathcal{P}_α -name is also a \mathcal{P}_β -name (literally).

¹²It follows of course that $q_1|_\alpha$ forces that $(f^{p_1,\alpha}, b^{p_1,\alpha}, \mathcal{O}^{p_1,\alpha})$ extends $(f^{p_0,\alpha}, b^{p_0,\alpha}, \mathcal{O}^{p_0,\alpha})$ in $\Theta_{\Phi^*(\alpha)}$ whenever $\alpha \in \text{dom}(p_0)$.

Note that if $\beta < \kappa$ is a nonzero limit ordinal and $q = (p, \Delta)$ satisfies condition (C0), then $q \in \mathcal{P}_\beta$ iff q satisfies (C1) and (C2). Note also that for all $\beta < \kappa$, \mathcal{P}_β is definable in $\langle H(\kappa), T^{\beta+1} \rangle$.

We will use the following easy lemma.

Lemma 2.5. *For all $\beta < \kappa$ and all $R \subseteq H(\kappa)$, if M is such that $\langle M, T^{\beta+1}, R \rangle \preceq \langle H(\kappa), T^{\beta+1}, R \rangle$, then \mathcal{P}_β forces $\langle M[\dot{G}_\beta], \dot{G}_\beta, R \rangle \preceq \langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, R \rangle$.*

Finally we give the definition of the forcing \mathcal{P}_κ . Conditions in \mathcal{P}_κ are pairs of the form $q = (p, \Delta)$ with the following properties.

- (E0) p is a finite function such that $\text{dom}(p) \subseteq \kappa$ and Δ is a set of pairs (N, γ) with $\gamma < \kappa$.
- (E1) For every $\alpha < \kappa$, the restriction of q to α is a condition in \mathcal{P}_α . This restriction is defined as

$$q|_\alpha := (p \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\})$$

Given conditions

$$q_\epsilon = (p_\epsilon, \Delta_\epsilon)$$

(for $\epsilon \in \{0, 1\}$) in \mathcal{P}_κ , we will say that $q_1 \leq_\kappa q_0$ if and only if the following holds.

- (F1) $q_1|_\alpha \leq_\alpha q_0|_\alpha$ for all $\alpha < \kappa$.

3. PROVING THEOREM 1.2: THE ACTUAL PROOF

In this section we prove the main facts about the forcings \mathcal{P}_α . Theorem 1.2 will follow immediately from them.

Our first lemma is immediate from the definitions.

Lemma 3.1. $\mathcal{P}_\kappa = \bigcup_{\beta < \kappa} \mathcal{P}_\beta$, and $\emptyset \neq \mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for all $\alpha \leq \beta \leq \kappa$.

Lemma 3.2 shows in particular that $\langle \mathcal{P}_\alpha : \alpha \leq \kappa \rangle$ is a forcing iteration in a broad sense.

Lemma 3.2. *Let $\alpha \leq \beta \leq \kappa$. If $q = (p, \Delta_q) \in \mathcal{P}_\alpha$, $s = (r, \Delta_s) \in \mathcal{P}_\beta$ and $q \leq_\alpha s|_\alpha$, then $(p^\frown(r \restriction [\alpha, \beta]), \Delta_q \cup \Delta_s)$ is a condition in \mathcal{P}_β extending s . Therefore, any maximal antichain in \mathcal{P}_α is a maximal antichain in \mathcal{P}_β and \mathcal{P}_α is a complete suborder of \mathcal{P}_β .*

Proof. It suffices to note that if the pair (N, γ) is in Δ_q , then the marker γ (which bounds the influence of the side condition N in clauses (C1) and (C3)) is at most α .

□

The following lemma shows that all forcings \mathcal{P}_β are \aleph_2 -Knaster, and so in particular have the \aleph_2 -chain condition.¹³ The proof uses standard Δ -system arguments¹⁴ (see Fact 2.3 and Lemma 2.4).

Lemma 3.3. *For every $\beta \leq \kappa$ and every set $\{(p_\xi, \Delta_\xi) : \xi < \omega_2\}$ of \mathcal{P}_β -conditions there is $I \subseteq \omega_2$ of size \aleph_2 such that for all ξ, ξ' in I :*

- (○) *if $\gamma \leq \beta$ and $\gamma < \kappa$, then $\Delta_\xi^{-1}(\gamma) \cup \Delta_{\xi'}^{-1}(\gamma)$ is a $\vec{T}^{\gamma+1}$ -symmetric $\mathcal{M}_*^{\gamma+1}$ -system of structures,*
- (○) *if $\alpha \in \text{dom}(p_\xi) \cap \text{dom}(p_{\xi'})$, $(f^{p_\xi, \alpha}, b^{p_\xi, \alpha}) = (f^{p_{\xi'}, \alpha}, b^{p_{\xi'}, \alpha})$ and $\mathcal{O}^{p_\xi, \alpha} \cup \mathcal{O}^{p_{\xi'}, \alpha}$ is a $\vec{T}^{\alpha+2}$ -symmetric $\mathcal{M}_*^{\alpha+2}$ -system of structures, and*
- (○) *letting $p^*(\alpha) = (f^{p_\xi, \alpha} \cup f^{p_{\xi'}, \alpha}, b^{p_\xi, \alpha} \cup b^{p_{\xi'}, \alpha}, \mathcal{O}^{p_\xi, \alpha} \cup \mathcal{O}^{p_{\xi'}, \alpha})$ for all $\alpha \in \text{dom}(p_\xi \cup p_{\xi'})$, $(p^*, \Delta_\xi \cup \Delta_{\xi'})$ is a \mathcal{P}_β -condition extending both (p_ξ, Δ_ξ) and $(p_{\xi'}, \Delta_{\xi'})$.¹⁵*

In particular, \mathcal{P}_β is \aleph_2 -Knaster.

Corollary 3.4. *If \dot{C} is a \mathcal{P}_α -name for a club-sequence, then \mathcal{P}_α forces $\Theta_{\dot{C}}$ to have the $(\aleph_2)^V$ -chain condition.¹⁶ Hence, every maximal antichain of $\Theta_{\dot{C}}$ is forced to be a member of $H(\kappa)$.*

Proof. Suppose \dot{A} is a \mathcal{P}_α -name for a maximal antichain of $\Theta_{\dot{C}}$ of size $(\aleph_2)^V$. For each $\zeta \in \omega_2$, let p_ζ be a \mathcal{P}_α -condition forcing $\dot{a}_\zeta = \check{b}_\zeta$ for some b_ζ in V , where \dot{a}_ζ denotes the ζ -th element of \dot{A} . By the above lemma, we may assume that all the p_ζ 's are pairwise \mathcal{P}_α -compatible. Using Fact 2.3 and Lemma 2.4, we can also assume that any common extension of p_ζ and $p_{\zeta'}$ forces that \check{b}_ζ and $\check{b}_{\zeta'}$ are $\Theta_{\dot{C}}$ -compatible, but that is a contradiction. \square

Strictly speaking, the following lemma will not be needed in the rest of the paper. However, its proof is a convenient warm-up for the proof of conclusion (2) $_\beta$ of Lemma 3.7.

Lemma 3.5. *Let $\beta < \kappa$, and suppose $q = (p, \Delta_q)$ is (M, \mathcal{P}_β) -generic whenever $q \in \mathcal{P}_\beta$ and $M \in \Delta_q^{-1}(\beta) \cap \mathcal{M}^{\beta+1}$.¹⁷ If \dot{C} is a \mathcal{P}_β -name for a club-sequence $(C_\delta)_{\delta < \omega_1}$, then \mathcal{P}_β forces that $\sigma_0 = (f_0, b_0, \mathcal{O}_0)$ is $(N[\dot{G}_\beta], \Theta_{\dot{C}})$ -generic whenever $\sigma_0 \in \Theta_{\dot{C}}$, $N \in \mathcal{O}_0 \cap \mathcal{M}^{\beta+2}$, $\delta_N \in \text{dom}(f_0)$, and $f_0(\delta_N) = \delta_N$.*

¹³A forcing P is μ -Knaster if every subset of P of cardinality μ includes a subset of cardinality μ of pairwise compatible conditions.

¹⁴This is the only place where we make use of CH.

¹⁵If α is not in the domain of p_ξ , then by $(f^{p_\xi, \alpha}, b^{p_\xi, \alpha}, \mathcal{O}^{p_\xi, \alpha})$ we will mean $(\emptyset, \emptyset, \emptyset)$ and similarly for $p_{\xi'}$.

¹⁶By lemma 3.7 we will see that $(\aleph_2)^V = (\aleph_2)^{V^{\mathcal{P}_\alpha}}$.

¹⁷We will see in Lemma 3.7 that this hypothesis is true.

Proof. Let us work in $V^{\mathcal{P}_\beta}$. Let $A \in N[\dot{G}_\beta]$ be a maximal antichain of $\Theta_{\dot{C}}$. By extending σ_0 if necessary we may assume that σ_0 extends a condition σ^\dagger in A . We want to show of course that σ^\dagger is in $N[\dot{G}_\beta]$, and for this it will suffice to find a member of $A \cap N[\dot{G}_\beta]$ compatible with σ_0 .

We may assume that $\delta_N \in \text{dom}(b)$, as otherwise the proof is slightly simpler. Let $\mu = \max(\text{range}(f_0 \upharpoonright \delta_N)) + 1$,¹⁸ let $\tilde{A} \in N[\dot{G}_\beta]$ be the (partially defined) function sending each $\sigma \in A$ to the first $\Theta_{\dot{C}}$ -condition (f', b', \mathcal{O}') extending σ (in some canonical well-ordering given by Φ) and such that $f_0 \upharpoonright \delta_N \subseteq f'$, $\text{range}(f') \cap \mu = \text{range}(f_0) \cap \mu$, $b_0 \upharpoonright \delta_N \subseteq b'$ and $\mathcal{O}_0 \cap N \subseteq \mathcal{O}'$ (whenever this is possible),¹⁹ and let $M \in N \cap \mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2}$ be such that $\tilde{A} \in M[\dot{G}_\beta]$, $\beta + 1 \in M$, and $\delta_M \notin C_{\delta_N}$. Such an M exists by condition (3.3) in the definition of $\Theta_{\dot{C}}$. Let $\eta < \delta_M$ be such that $[\eta, \delta_M] \cap C_{\delta_N} = \emptyset$ (this η exists by openness of $\delta_N \setminus C_{\delta_N}$). Now, in $M[\dot{G}_\beta]$ there is $\sigma_* = (f_*, b_*, \mathcal{O}_*)$ such that

- (a) $\sigma_* \in \text{range}(\tilde{A})$, and
- (b) $\min(\text{range}(f_*) \setminus \mu) > \eta$.

Note that $\max(\text{range}(f_*)) < \delta_M$ since $\delta_{M[\dot{G}_\beta]} = \delta_M$ by our assumption on $\mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}^{\beta+1}$. Let

$$\mathcal{O}' = \mathcal{O}_0 \cup \{\Psi_{N, N'}(M) : M \in \mathcal{O}_*, N' \in \mathcal{O}_0, \delta_{N'} = \delta_N\}$$

Now it is easy to check that $(f_* \cup f_0, b_* \cup b_0, \mathcal{O}')$ is a common extension of σ_* and σ_0 in $\Theta_{\dot{C}}$ (this uses condition (C1) in the definition of \mathcal{P}_β for the verification of conditions (1), (2.2) and (3.3) in the definition of $\Theta_{\dot{C}}$). Letting $\bar{\sigma} \in A \cap N[\dot{G}_\beta]$ be the unique $\sigma \in A$ such that $\tilde{A}(\sigma) = \sigma_*$,²⁰ it follows that $\bar{\sigma} = \sigma^\dagger$. \square

The following lemma can be proved easily by induction on α . It will be used in the proof of Lemma 3.7.

Lemma 3.6. *For all $\alpha < \kappa$, $v \in \mathcal{P}_\alpha$, and $\delta < \omega_1$ there is $w = (\bar{w}, \Delta_w) \in \mathcal{P}_\alpha$ extending v together with $\eta < \delta$ such that for all $M \in N$ and all $\xi \in \text{dom}(\bar{w}) \cap N$, if M and N are both in $\Delta_{w|_{\xi+1}}^{-1}(\xi+1)$, $\delta_N = \delta \in \text{dom}(b^{\bar{w}, \xi})$ and $b^{\bar{w}, \xi}(\delta_N) < \delta_M = f^{\bar{w}, \xi}(\delta_M)$, then $w|_\xi \Vdash_{\mathcal{P}_\xi} \dot{C}_{\delta_N}^\xi \cap [\eta, \delta_M] = \emptyset$.*

The properness of all \mathcal{P}_β ($\beta < \kappa$) is an immediate consequence of the following lemma.

¹⁸Note that $\text{range}(f_0 \upharpoonright \delta_N) \neq \emptyset$ by condition (3.1) in the definition of $\Theta_{\dot{C}}$.

¹⁹ \tilde{A} is in $N[\dot{G}_\beta]$ since $(f_0 \upharpoonright \delta_N, b_0 \upharpoonright \delta_N, \mathcal{O}_0) \in N$ and \tilde{A} is definable in the structure $\langle H(\kappa)^{V[\dot{G}_\beta]}, \dot{G}_\beta, T^{\beta+2} \rangle$ from $(f_0 \upharpoonright \delta_N, b_0 \upharpoonright \delta_N, \mathcal{O}_0)$ (by Lemma 2.5).

²⁰Note that \tilde{A} is one-to-one.

Lemma 3.7. *Suppose $\beta < \kappa$ and $N \in \mathcal{M}^{\beta+1}$. Then the following conditions hold.*

- (1) $_{\beta}$ *For every $q \in N \cap \mathcal{P}_{\beta}$ there is $q' \leq_{\beta} q$ such that $N \in \Delta_{q'}^{-1}(\beta)$.*
- (2) $_{\beta}$ *If $q \in \mathcal{P}_{\beta}$ and $N \in \Delta_q^{-1}(\beta)$, then q is (N, \mathcal{P}_{β}) -generic.*

Proof. The proof of (2) $_{\beta}$ will be the same in all cases, and the proof of (1) $_{\beta}$ will be by induction on β . The proof of (1) $_0$ is trivial: It suffices to set $q' = q \cup \{(N, 0)\}$.

The proof of (1) $_{\beta}$ when $\beta = \alpha + 1$ is as follows. Let $q = (p, \Delta_q)$. By (1) $_{\alpha}$ we may assume that there is a condition $t = (u, \Delta_t) \in \mathcal{P}_{\alpha}$ extending $q|_{\alpha}$ and such that $N \in \Delta_t^{-1}(\alpha)$. This condition t clearly forces (in \mathcal{P}_{α}) that $N \in \mathcal{N}_{\dot{G}_{\alpha}}$. So, t forces that for every $x \in N$, there is $M \in \mathcal{N}_{\dot{G}_{\alpha}} \cap \mathcal{M}^{\alpha+1}$ such that $x \in M$.

Let us work in $V^{\mathcal{P}_{\alpha}[t]}$. Since, by Lemma 2.5, $\langle N[\dot{G}_{\alpha}], \dot{G}_{\alpha}, T^{\alpha+2}, H(\kappa)^V \rangle$ is an elementary substructure of $\langle H(\kappa)[\dot{G}_{\alpha}], \dot{G}_{\alpha}, T^{\alpha+2}, H(\kappa)^V \rangle$, there exists an M as above in $N[\dot{G}_{\alpha}] \cap V$ (where V denotes the ground model). We can also assume that $M \in N$, since $N[\dot{G}_{\alpha}] \cap V = N$ (which follows from (2) $_{\alpha}$ applied to N and t). This shows that t forces $\text{rank}(\mathcal{N}_{\dot{G}_{\alpha}} \cap \mathcal{M}^{\alpha+1}, N) \geq 1$. In fact, a similar argument shows that t forces $\text{rank}(\mathcal{N}_{\dot{G}_{\alpha}} \cap \mathcal{M}^{\alpha+1}, N) > \mu$ for every $\mu < \delta_N$. In view of these considerations, it suffices to define q' as the condition $(u', \Delta_q \cup \Delta_t \cup \{(N, \beta)\})$, where u' extends u and sends the ordinal α to the triple $(f^{p,\alpha} \cup \{\langle \delta_N, \delta_N \rangle\}, b^{p,\alpha}, \mathcal{O}^{p,\alpha} \cup \{N\})$.

The proof of (1) $_{\beta}$ when β is a nonzero limit ordinal is trivial using (1) $_{\alpha}$ for all $\alpha < \beta$, together with the fact if $q = (p, \Delta) \in \mathcal{P}_{\beta}$, then the domain of p is bounded in β .

Now let us proceed to the proof of (2) $_{\beta}$ for general β . Let $A \subseteq \mathcal{P}_{\beta}$ be a maximal antichain in N , and suppose $q = (p, \Delta_q)$ extends a condition in A . We want to see that q is compatible with a condition in $A \cap N$.

Let ξ_0 be the maximum of the set X of $\xi \in \text{dom}(p)$ such that $\xi \in N'$ for some pair $(N', \gamma) \in \Delta_q$ with $\xi < \gamma$ and $\delta_{N'} = \delta_N$. Let $(N', \gamma) \in \Delta_q$ witness $\xi_0 \in X$. By extending q further if necessary, we may assume that there is some $M' \in N'$ such that $(M', \xi_0) \in \Delta_q$ and such that $\Psi_{N,N'}(x) \in M'$ for all relevant $x \in N$. Let $M = \Psi_{N',N}(M')$. If $\delta_N \in \text{dom}(b^{p,\xi_0})$, we may assume that there is $\eta < \delta_M$ such that $q|_{\xi_0}$ forces $\dot{C}_{\delta_N}^{\xi_0} \cap [\eta, \delta_M] = \emptyset$. By Lemma 3.6 we may further assume that $q|_{\xi}$ forces $\dot{C}_{\delta_N}^{\xi} \cap [\eta, \delta_M] = \emptyset$ whenever $\xi \in X$, $\delta_N = \delta_{\overline{N}} \in \text{dom}(b^{p,\xi})$, $\overline{M} \in \overline{N}$, $\delta_{\overline{M}} = \delta_M$, $\overline{M}, \overline{N}$ are both in $\Delta_{q|_{\xi+1}}^{-1}(q|_{\xi+1})$, and $b^{p,\xi}(\delta_N) < \delta_M = f^{p,\xi}(\delta_M)$.

Claim 3.8. *Let $\xi \in X$.*

- (a) If $\xi = \xi_0$ and $\delta_N \in \text{dom}(b^{p,\xi_0})$, then $q|_{\xi_0}$ forces $\dot{C}_{\delta_N}^{\xi_0} \cap [\eta, \delta_M] = \emptyset$.
- (b) If $\xi \neq \xi_0$, $(\bar{N}, \bar{\gamma}) \in \Delta_q$, $\delta_N = \delta_{\bar{N}}$, $\xi \in \bar{N} \cap \bar{\gamma} \cap \Psi_{N,\bar{N}}(M)$ and $\delta_N \in \text{dom}(b^{p,\xi})$, then $(\Psi_{N,\bar{N}}(M), \xi + 1) \in \Delta_{q|_{\xi+1}}$ and δ_M is a fixed point of $f^{p,\xi}$. In particular, these hypothesis imply that $q|_{\xi}$ forces $\dot{C}_{\delta_N}^{\xi} \cap [\eta, \delta_M] = \emptyset$.

Proof. In order to prove (b), it is enough to note that $\Psi_{N,\bar{N}}(M) = \Psi_{N',\bar{N}}(M')$ and to apply clauses (C1) and (C3.3) to condition $q|_{\xi+1}$. \square

Let $\{\delta_0, \dots, \delta_{l-1}\} = \{\delta_{N'} : N' \in \Delta_{q|_0}^{-1}(0)\} \cap \delta_M$. By correctness of M and since M contains all relevant objects, there is a condition $t = (\bar{t}, \Delta_t) \in M$ satisfying the following properties (in V).

- (1) $t \in A$.
- (2) For all W in $\Delta_{q|_0}^{-1}(0) \cap M$ and for all $\zeta \in \beta + 1$, if $\zeta \in W$ and $W \in \Delta_{q|_{\zeta}}^{-1}(\zeta)$, then $W \in \Delta_{t|_{\zeta}}^{-1}(\zeta)$.²¹
- (3) For all $\zeta \in \text{dom}(\bar{t})$, if $\zeta \in \text{dom}(p)$, then
 - (3.1) $(f^{\bar{t},\zeta}, b^{\bar{t},\zeta}, \mathcal{O}^{\bar{t},\zeta})$ and $(f^{p,\zeta}, b^{p,\zeta}, \mathcal{O}^{p,\zeta})$ are forced by $q|_{\zeta}$ to be compatible as $\Theta_{\Phi^*(\zeta)}$ -conditions, and
 - (3.2) the least point in $\text{dom}(f^{\bar{t},\zeta})$ above $\text{dom}(f^{p,\zeta})$ is above η .
- (4) For all $i < l$, for all $\xi \in \text{dom}(p)$ and for all $(N', \gamma) \in \Delta_q$ with $\xi < \gamma$ and $\delta_{N'} = \delta_N$, if there is no W such that $\xi \in W$, $W \in \Delta_{q|_{\xi+1}}^{-1}(\xi + 1)$ and $\delta_W = \delta_i$, then letting $\eta = \Psi_{N',N}(\xi)$ and $\rho = \min((OR \cap M) \setminus \xi)$, there is no W such that $\eta \in W$, $W \in \Delta_{t|_{\rho}}^{-1}(\rho)$ and $\delta_W = \delta_i$.²²
- (5) If $(M', \gamma) \in \Delta_t$ and $\delta_{M'} \notin \{\delta_0, \dots, \delta_{l-1}\}$, then $\delta_{M'} > \eta$.

Using Claim 3.8, it is easy to check that one can amalgamate q and t into a condition q^* with Δ_{q^*} being the union of Δ_q with the set of all pairs $(\Psi_{N,N'}(W), \min\{\rho, \gamma\})$ such that $(W, \rho) \in \Delta_t$, $(N', \gamma) \in \Delta_q$, and $\delta_{N'} = \delta_N$. The closure of Δ_{q^*} under isomorphisms does not interfere with the elements of X because of condition (4). \square

Corollary 3.9. *For every $\beta \leq \kappa$, \mathcal{P}_β is proper.*

²¹Note that for such a W , the set of those ζ in $M \cap (\beta + 1)$ such that $\zeta \in W$ and $W \in \Delta_{q|_{\zeta}}^{-1}(\zeta)$ can be correctly computed (in M) by means of a formula using as parameters the structure W and the minimum ordinal in M which is at least the maximum of those ς such that $(W, \varsigma) \in \Delta_q$.

²²By clause (C1) applied to condition $q|_{\xi+1}$, the existence of a W such that $\Psi_{N',N}(\xi) \in W \in \Delta_{t|_{\rho}}^{-1}(\rho)$ would imply that $\xi = \Psi_{N,N'}(\Psi_{N',N}(\xi)) \in \Psi_{N,N'}(W) \in \Delta_{q|_{\xi+1}}^{-1}(\xi + 1)$.

Proof. For $\beta < \kappa$ the conclusion follows immediately from Lemma 3.7. The remaining case follows from the corresponding conclusions for $\beta < \kappa$ together with the \aleph_2 -c.c. of \mathcal{P}_κ and $cf(\kappa) \geq \omega_2$. \square

For every $\beta < \kappa$ let \dot{F}_β and \dot{B}_β be \mathcal{P}_κ -names for, respectively, the union of all functions f for which there is a condition $q = (p, \Delta) \in \dot{G}_\kappa$ such that $p(\beta) = (f, b, \mathcal{O})$ for some b and \mathcal{O} , and the union of all b for which there is a condition $q = (p, \Delta) \in \dot{G}_\kappa$ such that $p(\beta) = (f, b, \mathcal{O})$ for some f and \mathcal{O} .

Lemma 3.10. \mathcal{P}_κ forces $2^{\aleph_0} = \kappa$.

Proof. In order to prove that \mathcal{P}_κ forces $2^{\aleph_0} \geq \kappa$, it suffices to note that if $\beta < \kappa$, then the restriction of \dot{F}_β to ω is forced to be a Cohen real (recall that \dot{F} is a name for a normal function having ω as a fixed point). The other inequality follows from counting nice names for subsets of ω using the fact that $\kappa^{\omega_1} = \kappa$ together with Lemma 3.3. \square

Lemma 3.11. For all $\beta < \kappa$, \mathcal{P}_κ forces that $\text{range}(\dot{F}_\beta)$ is a club of ω_1 measuring $\Phi^*(\beta)$.

Proof. Let \dot{C}_δ be, for each δ , a \mathcal{P}_β -name for the δ -th member of $\Phi^*(\beta)$. We want to show that the following conditions hold in $V^{\mathcal{P}_\kappa}$:

- (A) \dot{F}_β is a normal function with domain ω_1 .
- (B) For each $\nu < \omega_1$,
 - (B1) if $\nu \in \text{dom}(\dot{B}_\beta)$, then $\text{range}(\dot{F}_\beta \restriction (\dot{B}_\beta(\nu), \nu))$ is disjoint from $\dot{C}_{\dot{F}(\nu)}$, and
 - (B2) if $\nu \notin \text{dom}(\dot{B}_\beta)$, then a tail of $\text{range}(\dot{F}_\beta \restriction \nu)$ is included in $\dot{C}_{f(\nu)}$.

Showing (A) is easy, so here we will only show (B). Note that for every $q = (p, \Delta) \in \mathcal{P}_\kappa$, if $p(\beta) = (f, b, \mathcal{O})$ and $\nu \in \text{dom}(f)$, then there is some $q' = (p', \Delta')$ extending q such that, letting $p'(\beta) = (f', b', \mathcal{O}')$, either

- (a) $\nu \in \text{dom}(b')$, or
- (b) q' forces $\text{rank}(\{M \in \mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2} : \delta_M \notin \dot{C}_{f(\nu)}\}, N) = \nu'$ for some given $\nu' < \nu$ for every (equivalently, for some) $N \in \mathcal{O}'$ such that $\delta_N = f(\nu)$.

It is enough to assume (b) and show that q' forces that a tail of $\text{range}(\dot{F}_\beta \restriction \nu)$ is included in $\dot{C}_{f(\nu)}$. For this, fix an N as in (b) and, extending q' if necessary, fix also $x \in N$ such that $q'|_\beta$ forces that if $M \in N$ is such that $x \in M$ and $\text{rank}(\mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2}, M) > \nu'$, then $\delta_M \in \dot{C}_{f(\nu)}$. By further extending q' if necessary we may assume that

$x \in M$ for some $M \in \Delta_{q'}^{-1}(\beta) \cap \mathcal{M}_*^{\beta+2} \cap N$ such that $x \in M$ and $\delta_M = f(\nu_*)$ for some $\nu_* \geq \nu'$ below ν . Now suppose $q'' = (p'', \Delta_{q''})$ extends q' and suppose $\nu_o \in \text{dom}(f^{p'',\beta})$ is in $[\nu_*, \nu)$. It suffices to show that $q''|_\beta$ forces $f^{p'',\beta}(\nu_o) \in \dot{C}_{f(\nu)}$.

For this, note that $q''|_\beta$ forces that $f^{p'',\beta}(\nu_o)$ is δ_{M_o} for some $M_o \in \mathcal{O}^{p'',\beta} \subseteq \mathcal{M}_*^{\beta+2} \cap \mathcal{N}_{\dot{G}_\beta}$ such that $\text{rank}(\mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2}, M_o) \geq \nu_o$. By symmetry of $\mathcal{O}^{p'',\beta}$ ²³ and since $\delta_{M_o} > \delta_M$ there is then some $M'_o \in \mathcal{O}^{p'',\beta}$ such that $M \in M'_o$ and $\delta_{M'_o} = \delta_{M_o}$. Since, by symmetry, $q''|_\beta$ forces $\text{rank}(\mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2}, M'_o) = \text{rank}(\mathcal{N}_{\dot{G}_\beta} \cap \mathcal{M}_*^{\beta+2}, M_o) \geq \nu_o$ and since $x \in M'_o$, it follows that $q''|_\beta$ forces $f^{p'',\beta}(\nu_o) = \delta_{M'_o} \in \dot{C}_{f(\nu)}$, which is what we wanted. \square

Corollary 3.12. \mathcal{P}_κ forces measuring.

The above corollary follows from Lemmas 3.11, 3.3 and 3.7 (since Φ was chosen to be a book-keeping function), and finishes the proof of Theorem 1.2.

²³Specifically, by condition (γ) in the definition of symmetric system.

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